4 VORTICES

VORTICES ARE CO-DIMENSION 2 SOLITONS; PARTICLES IN 2+1, STRINGS IN 3+1.

ABELIAN-HIGGS MODEL

$$S = \int d^3x\, dt \left(-\frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \overline{D_\mu \phi}\, D^\mu \phi - \frac{\lambda}{2}\left(v^2 - |\phi|^2\right)^2 \right)$$

IN THIS MODEL WE FIND THE SO CALLED

$\underbrace{\text{Abrikosov}}_{\text{Superconductors}}$-$\underbrace{\text{Nielsen-Olesen}}_{\text{Relativistic String}}$ (ANO) VORTEX

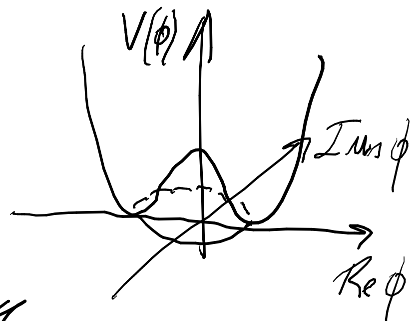
ϕ COMPLEX SCALAR FIELD

$$\phi \to e^{i\alpha(x)}\phi\,, \quad a_\mu \to a_\mu + \partial_\mu \alpha(x)$$

$$D_\mu \phi = \partial_\mu \phi - i e a_\mu \phi$$ COVARIANT DERIVATIVE.

IT IS JUST SCALAR ELETTRO-DYNAMICS WITH A SYMMETRY BREAKING

"MEXICAN HAT" POTENTIAL

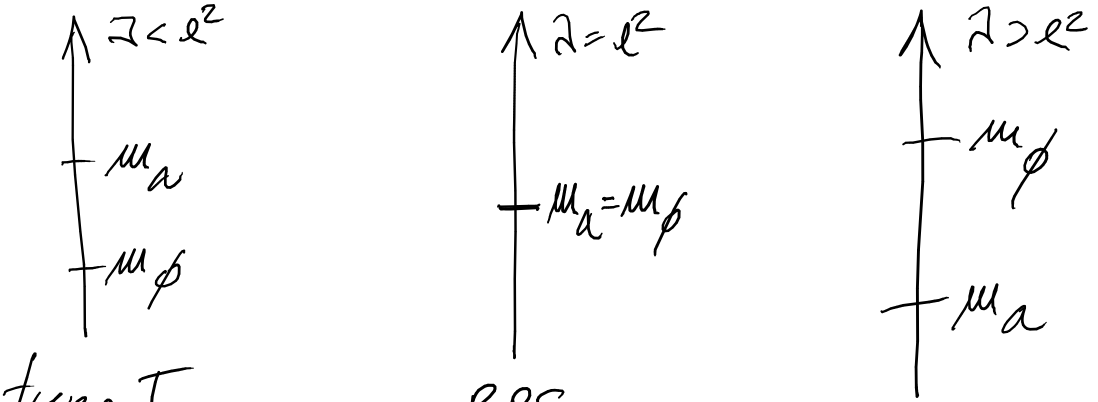


$\langle \phi \rangle = v e^{i\alpha}$ VACUUM

$m_\phi = \sqrt{2\lambda}\, v$ THIS IS THE "HIGGS" FIELD AND CORRESPOND TO THE RADIAL FLUCTUATION OF ϕ

$m_{a_\mu} = \sqrt{2}\, e v$ "HIGGS MECHANISM"

NOTE THE MASS GAP



type I BPS type II

FOR CONFIGURATIONS THAT ARE NOT DEPENDENT ON TIME t AND z (x_3), WE HAVE

$$E = \int d^2x \left(\frac{B^2}{2} + \overline{D_i\phi}\, D_i\phi + \frac{\lambda}{2}\left(v^2 - |\phi|^2\right)^2 \right)$$

THIS IS THE GINSBURG-LANDAU ENERGY

$$B = f_{12} = \partial_1 a_2 - \partial_2 a_1$$

E-L EQUATIONS ARE (FOR STATIC):

$$\begin{cases} D_i D_i \phi + \lambda\left(v^2 - |\phi|^2\right)\phi = 0 & \Leftarrow \dfrac{\delta}{\delta\bar\phi} \\ \epsilon_{ij}\partial_j B + i\left(\bar\phi D_i\phi - \phi D_i\bar\phi\right) & \Leftarrow \dfrac{\delta}{\delta a_i} \end{cases}$$

THE LAST IS JUST THE AMPÈRE EQUATION $\vec\nabla \wedge \vec B = \vec J$

THE VORTEX IS A TOPOLOGICAL SOLITON SOLUTION OF THESE EQUATIONS.

FOR 1-SOLITON STATE IS VERY CONVENIENT TO USE POLAR COORDINATES ρ, ϑ

$$E = \int_0^\infty \rho\, d\rho \int_0^{2\pi} d\vartheta \cdot \left(\frac{1}{2\rho^2} f_{\rho\vartheta}^2 + \overline{D_\rho \phi}\, D_\rho \phi + \frac{1}{\rho^2} \overline{D_\vartheta \phi}\, D_\vartheta \phi + \frac{\lambda}{2} \left(v^2 - |\phi|^2\right)^2 \right)$$

$f_{\rho\vartheta} = \rho B$

MOREOVER WE CAN CHOSE THE RADIAL GAUGE $a_\rho = 0$

SO THE FIELDS WE HAVE TO DETERMINE ARE a_ϑ, ϕ AS FUNCTION OF ρ, ϑ.

FINITE ENERGY CONDITION:

$$\lim_{\rho \to \infty} \phi(\rho, \vartheta) = v e^{i \chi^\infty(\vartheta)}$$

χ IS A CERTAIN PHASE

THIS TO HAVE $V(\phi) \to 0$

MOREOVER WE WANT $D_\vartheta \phi \to 0$, $f_{\rho\vartheta} \to 0$

$$\lim_{\rho \to \infty} a_\vartheta(\rho, \vartheta) = a_\vartheta^\infty(\vartheta)$$

$\partial_\rho a_\vartheta \to 0$ SUFFICIENTLY FAST

$a_\vartheta^\infty(\vartheta) = \frac{1}{e} \partial_\vartheta \chi^\infty(\vartheta)$ SO THAT $D_\vartheta \phi \to 0$

$\hookrightarrow$ THIS GIVES MORE FREEDOM RESPECT TO PURELY SCALAR FIELD THEORY

$\chi^\infty(\vartheta)$ <u>DOES NOT</u> HAVE TO BE CONSTANT!

TOPOLOGY CLASSIFICATION

$$\phi^{\infty} : S^1_{\infty} \longrightarrow S^1$$

$\rho \to \infty$ $\qquad\qquad$ $V = 0$

TOPOLOGICAL NUMBER

$$N = \frac{1}{2\pi}\int_0^{2\pi} \partial_\theta \chi^{\infty} \, d\theta = \frac{1}{2}\left(\chi^{\infty}(2\pi) - \chi^{\infty}(0)\right)$$

WINDING NUMBER. $\qquad \in \pi_1(S^1) = \pi_1(U(1))$

THIS IS ALSO THE FIRST CHERN NUMBER

$$\frac{1}{2\pi}\int a^{\infty}_{\theta}(\theta)\, d\theta = \frac{1}{2\pi}\int_{\mathbb{R}^2} B \, d^2x = \frac{\Phi_B}{2\pi}$$

WE ARE ASSURED THAT THIS IS A GAUGE INVARIANT QUANTITY.

$\Phi_B = \frac{2\pi n}{e}$ $\qquad\qquad$ $\pi_1(S^1)$

1st Chern Number $\qquad\qquad$ SCALAR FIELD

FINITE ENERGY CONDITION

NOW WE WRITE AN "ANSATZ" FOR THE FIELD CONFIGURATION IN THIS TOPOLOGICAL SECTOR:

$\phi = e^{in\theta} f(\rho)$, $a_\theta(\rho)$ $\quad$ 2 REAL FUNCTIONS

BOUNDARY CONDITION

$\rho \to \infty$ $\quad f(\rho) \to v \quad a_\theta(\rho) \to \frac{n}{e}$

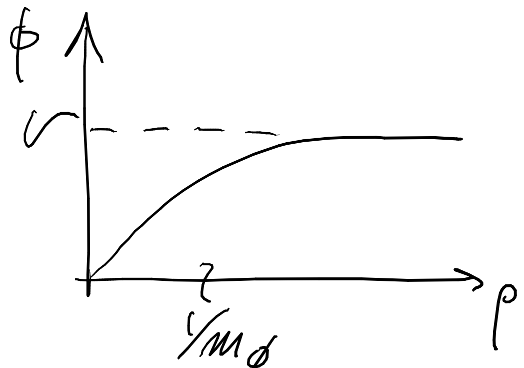
FROM FINITE ENERGY CONDITION

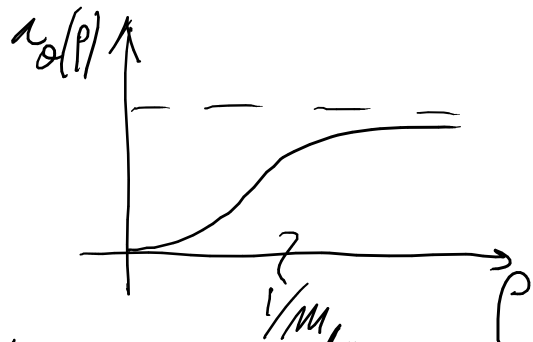
$\rho \to 0$ $\quad \underbrace{f(\rho) \to 0 \quad a_\theta(\rho) \to 0}_{\text{CONTINUITY}}$

MORE PRECISELY FROM E-L

$f(\rho) \propto \rho^n \qquad a_\theta(\rho) \propto \rho^2$

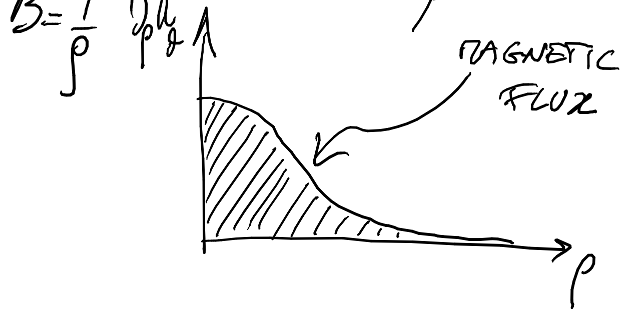
FOR $n=1$





$$\Phi = 2\pi \int_0^\infty \rho \, d\rho \, \frac{\partial_\rho a_\theta}{\rho}$$

$$= \frac{2\pi n}{e}$$



$U(1) \to 1$

$G \to H$ $\qquad \pi_1(G/H) = \pi_1(U(1)) = \mathbb{Z}$

WE CAN MAKE FURTHER PROGRESS IF WE RESTRICT TO THE CASE $\lambda = e$ (CRITICAL COUPLING, $m_\phi = m_{a_\mu}$). IN THIS CASE WE CAN USE THE BOGOMOLNY MACHINERY.

LET'S RETURN TO x_i COORDINATES

$$E = \int \left(\frac{B^2}{2} + \overline{D_i \phi}\, D_i \phi + \frac{e^2}{2}\left(v^2 - \bar{\phi}\phi\right)\right) d^2x$$

SET $\lambda = e^2$ BPS CASE

BOGOMOLNY DECOMPOSITION:

$$E = \int d^2x \left\{ \frac{1}{2}\left(B - e\left(v^2 - \bar{\phi}\phi\right)\right)^2 + \left(\overline{D_1\phi} - i\overline{D_2\phi}\right)\left(D_1\phi + iD_2\phi\right) \right.$$
$$\left. + e\,B\left(v^2 - \bar{\phi}\phi\right) - i\,\overline{D_1\phi}\, D_2\phi + i\, D_1\phi\, \overline{D_2\phi} \right\}$$

THE STRATEGY IS THE SAME USED BEFORE FOR THE KINK

$$E = \int d^dx \left\{ \underbrace{(\quad)^2 + \dots + (\quad)^2}_{\text{ALWAYS} \geq 0} + \underbrace{\text{something}}_{} \right\}$$

NOW WE HAVE TO MANIPULATE A BIT THE EXTRA TERM

$$\text{something} = \int d^2x \left\{ eBv^2 - e\bar{\phi}\left(\partial_1 a_2 - \partial_2 a_1\right)\phi - i\left(\partial_1 + iea_1\right)\bar{\phi}\left(\partial_2 - iea_2\right)\phi \right.$$
$$\left. + i\left(\partial_1 - iea_1\right)\phi\left(\partial_2 + iea_2\right)\bar{\phi} \right\}$$

$$= \int d^2x \, \Big\{ eBv^2 - e\bar\phi\,(\partial_1 a_2 - \partial_2 a_1)\phi - i\partial_1\bar\phi\,\partial_2\phi + i\partial_1\phi\,\partial_2\bar\phi$$
$$- e\partial_1\bar\phi\, a_2\phi - e\partial_1\phi\, a_2\bar\phi + e\partial_2\bar\phi\, a_1\phi + e\partial_2\phi\, a_1\bar\phi \Big\}$$

$$= \int d^2x \Big\{ eBv^2 + e\,\partial_1(-\bar\phi a_2\phi) + e\,\partial_2(\bar\phi a_1\phi)$$
$$- i\,\partial_1(\bar\phi\,\partial_2\phi) + i\,\partial_2(\phi\,\partial_1\phi) \Big\}$$

$$= \int d^2x \Big\{ eBv^2 \underbrace{- i\,\partial_1(\bar\phi D_2\phi) + i\,\partial_2(\bar\phi D_1\phi)}_{\text{BOUNDARY TERM, BUT } D\phi\to 0 \text{ SO DOES NOT CONTRIBUTE}} \Big\}$$

$$= \int d^2x\, eBv^2 = 2\pi n v^2$$

BOGOMOLNY BOUND IS $2\pi|n|v^2$ (REMEMBER THAT FOR $N<0$ WE HAVE TO FLIP SOME SIGNS...).

THE BOUND IS SATURATED IF THE BOGOMOLN'Y EQUATIONS ARE SATISFYED

$$\begin{cases} D_1\phi + i D_2\phi = 0 \\ B - e(v^2 - \bar\phi\phi) = 0 \end{cases}$$

1^{st} ORDER EQUATION.

5) MONOPOLES

ALL BEGAN WITH THE DIRAC MONOPOLE...

Maxwell's equations:

$$\vec{\nabla}\cdot\vec{E} = \rho_e \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\wedge\vec{E} = -\frac{\partial B}{\partial t} \qquad \vec{\nabla}\wedge\vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{J}_e$$

$\vec{\nabla}\cdot\vec{B} = 0 \Longrightarrow$ THERE IS NO SOURCE FOR THE MAGNETIC FIELD

WITHOUT ELECTRIC SOURCES THE EQUATIONS ARE INVARIANT UNDER

$$\vec{E} + i\vec{B} \longrightarrow e^{i\alpha}(\vec{E} + i\vec{B})$$

$\alpha = \pi/2 \qquad \vec{E} \to -\vec{B} \qquad \vec{B} \to \vec{E}$ "S_DUALITY"

OR

"ELETTRO_MAGNETIC DUALITY"

$(\rho_e, \vec{J}_e)$ BREAK THE DUALITY. THIS COULD BE RESTORATED BY ADDING MAGNETIC SOURCES $(\rho_m, \vec{J}_m)$. IN RELATIVISTIC FORMULATION

$$f_{\mu\nu} \qquad {}^*f_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} f_{\rho\sigma}$$

$$\begin{cases} \partial_\mu f^{\mu\nu} = J^{\nu(e)} \\ \partial_\mu {}^*f^{\mu\nu} = J^{\nu(m)} \end{cases}$$

FOR CLASSICAL ELETTRO_DYNAMICS THERE IS NO THEORETICAL OBSTRUCTION IN INTRODUCING THE MAGNETIC SOURCES.
THE ONLY INCONVENIENT IS THAT WE CANNOT INTRODUCE A GAUGE POTENTIAL

IF $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \Rightarrow \partial_\mu {}^{*}f_{\mu\nu} = 0$

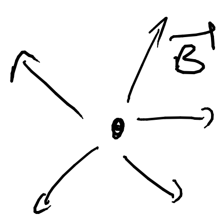
IN QUANTUM MECHANICS a_μ HAS INSTEAD A PHYSICAL EFFECT (EVEN WHEN $f_{\mu\nu}=0$).
SO WE MAY SUSPECT THAT
MONOPOLES + Q.M. WILL LEAD TO SOME PROBLEMS (OR SOMETHING INTERESTING).

Dirac Monopole

WE INSIST ON HAVING BOTH MONOPOLE AND GAUGE POTENTIAL a_μ. WE HAVE TO GIVE UP SOMETHING: REGULARITY EVERYWHERE.

$$\vec{B} = \frac{g}{4\pi r^2}\hat{r} \qquad \int d\vec{S}\cdot\vec{B} = g = \text{MAGNETIC CHARGE}$$

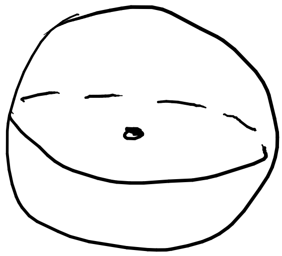
$$\vec{\nabla}\cdot\vec{B} = g\,\delta^3(\vec{r})$$



$$\rho^{(m)} = g\,\delta^3(\vec{r})$$

$$\vec{B} = \vec{\nabla}\wedge\vec{a}$$

WE WANT TO FIND $\vec{a}$ THAT SOLVES THIS EQUATION



$$\int \vec{B} \cdot d\vec{s} = g = \int \vec{\nabla} \wedge \vec{a} \cdot d\vec{s}$$

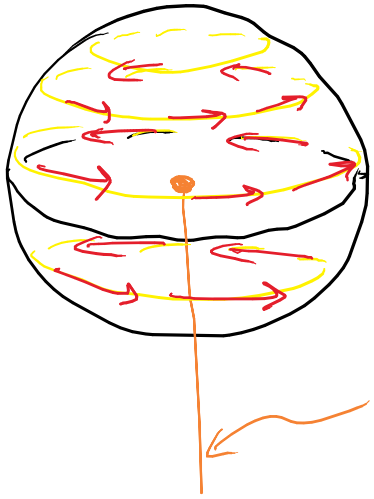
$\forall\ S^2$ SPHERE AROUND THE SOURCE

THE SPHERE MUST BE PINCHED BY SOME SINGULARITY

DIRAC SOLUTION:

$$\vec{a} = \frac{g}{4\pi r} \frac{1-\cos\theta}{\sin\theta} \vec{l_\varphi}$$

$\vec{l_\varphi}$ IS THE UNIT VECTOR IN φ DIRECTION



$\vec{a}$ ORIENTED ALONG $\vec{l_\varphi}$

"DIRAC STRING"

PLACE WHER a DIVERGES.

$$\vec{\nabla} \wedge \vec{a} = \frac{g\vec{r}}{4\pi r^2}$$

SPHERICAL SYMMETRIC NON SINGULAR

NON SPHERICAL SYMMETRIC AND SINGULAR

BOTH ARE GAUGE ARTIFACTS, BUT UNAVOIDABLES!

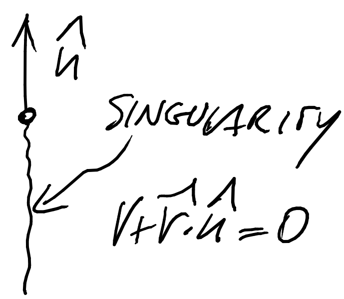
IN CARTESIAN COORDINATES

$$\vec{a} = \frac{g}{4\pi r}\,\frac{(-y, x, 0)}{r+z} \qquad \left(\frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \frac{g}{4\pi r}\,\frac{(0,0,1)\wedge(x,y,z)}{r+z} \qquad \hat{n} = (0,0,1)$$

WE CAN WRITE A DIFFERENT GAUGE POTENTIAL FOR ARITRARY CHOICE OF $\hat{n}$

$$\vec{a}(\hat{n}) = \frac{g}{4\pi r}\,\frac{\hat{n}\wedge\vec{r}}{(r+\vec{r}\cdot\hat{n})}$$

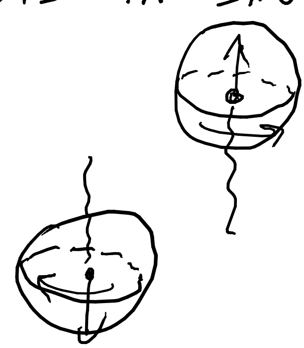


$$\vec{\nabla}\wedge\vec{a}(\hat{n}) = \frac{g\,\vec{r}}{4\pi r^2} \quad \forall \text{ CHOICE OF } \hat{n}$$

WE SEE THAT THE SINGULARITY CAN BE MOVED.

TWO PATCHES CAN BE USED TO COVER THE ENTIRE SPACE AND AVOID THE SINGULARITY

$$\vec{a}_N = \frac{g}{4\pi r}\,\frac{(-y, x, 0)}{r+z}$$



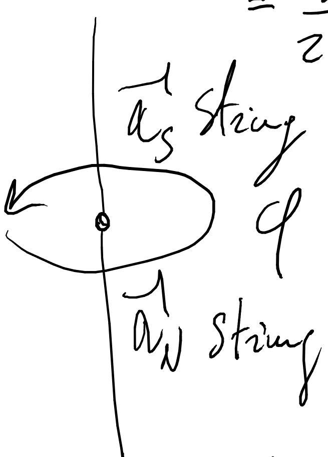
$$\vec{a}_S = \frac{g}{4\pi r}\,\frac{(y, -x, 0)}{r-z}$$

WU-YANG MONOPOLE (CONNECTION WITH FIBRE BUNDLES).

$$\vec{a}_N - \vec{a}_S = \frac{g}{4\pi r}\left(\frac{1}{r+z} + \frac{1}{r-z}\right)(-y, x, 0)$$

$$= \frac{g}{4\pi r}(-y, x, 0)\frac{2r}{r^2 - z^2}$$

$$= \frac{g}{2\pi}(-y, x, 0)\frac{1}{x^2+y^2} = \frac{g}{2\pi}\vec{\nabla}\left(\varphi = \arctan\frac{y}{x}\right)$$



GAUGE TRANSFORMATION

NOW LET'S CONSIDER A "PROBE" CHARGED PARTICLE ψ IN THE BACKGROUND OF THE MONOPOLE. $D_i\psi = \partial_i\psi - i\, a_i \psi$

IN Q.M. THE Aharonov-Bohm PHASE IS A PHYSICAL OBSERVABLE, UNLESS

IF $\int \vec{a} = 2\pi n$, WHERE $n \in \mathbb{Z}$

NEAR THE SINGULARITY

$\vec{a}_N \simeq \vec{\nabla}\alpha$, $\alpha = \frac{g}{2\pi}\varphi$

SO, FOR THE DIRAC STRING TO BE UNOBSERVABLES,

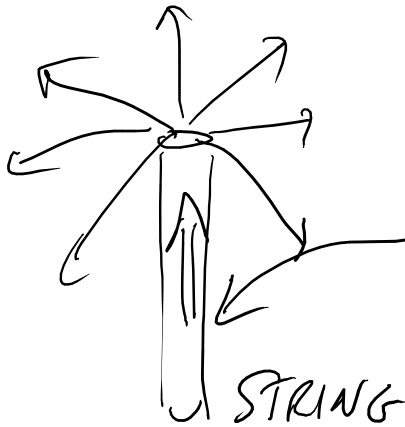
$$\boxed{g = 2\pi N}$$

FOR WU-YANG MONOPOLE

$$\vec{a}_N - \vec{a}_S = \frac{g}{2\pi}\vec{\nabla}\varphi$$

$$\psi \to e^{i g \frac{\varphi}{2\pi}}\psi \quad \text{GAUGE TRANSFORMATION}$$

FOR ψ TO BE SINGLE VALUED IN BOTH PATCHES, WE NEED $g = 2\pi n$. SAME CONDITION OBTAINED WITH THE A-B EFFECT.



INFINITESIMAL STRING IS UNOBSERVABLE FOR $g = 2\pi n$

DIRAC INTERPRETATION: PRESENCE OF MONOPOLE IMPLIES THE CHARGE QUANTIZATION IN INTEGER VALUES

$$eg = 2\pi n$$

ELECTRIC CHARGE $D_\mu = \partial_\mu - ieA_\mu$

DIRAC MONOPOLE IS NOT A PROPER SOLUTION

$$E = \frac{1}{2}\int B^2 \sim \frac{g^2}{8\pi}\int d\rho \frac{1}{\rho^2}$$

UV DIVERGENT.

't HOOFT-POLYAKOV MONOPOLE

THIS IS A REAL SOLITON: FINITE ENERGY, TOPOLOGICALLY STABLE SOLUTION WITH MAGNETIC CHARGE.

YANG-MILLS + HIGGS THEORY

THE PROTOTYPE MODEL IS THE SO CALLED GEORGI-GLASHOW MODEL, $SU(2)$ YM + ϕ HIGGS FIELD IN THE ADJOINT REPRESENTATION (TRIPLET)

$$SU(2) \xrightarrow{\langle\phi\rangle} U(1)$$ VACUUM BREAKS GAUGE GROUP TO $U(1)$.

STANDARD MODEL $SU(2)\times U(1) \xrightarrow{\langle\phi\rangle} U(1)$ DOES NOT ADMIT MAGNETIC MONOPOLES.

GRAND UNIFICATION THEORY

$$SU(5) \xrightarrow{\langle\phi\rangle} \underbrace{SU(3)\times SU(2)\times U(1)}_{\text{SM GROUP}}$$

THIS ADMITS MONOPOLES AT THE GUT SCALE ($\simeq 10^{16}$ GeV).

Pauli Matrices $\sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

$t^a = \dfrac{\sigma^a}{2}$ GENERATORS LIE ALGEBRA $su(2)$

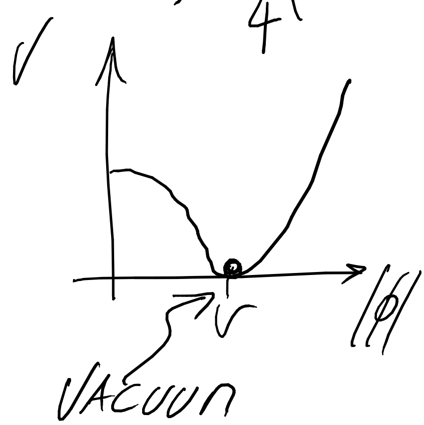
$$\begin{cases} [t^a, t^b] = i\epsilon^{abc} t^c \\ \mathrm{tr}(t^a t^b) = \dfrac{\delta^{ab}}{2} \end{cases}$$

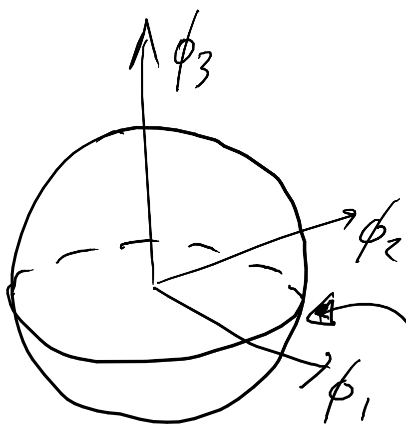
$D_\mu = \partial_\mu - ig[A_\mu, \ ]$

$\phi = \phi^a t^a$, $A_\mu = A_\mu^a t^a$ ADJOINT FIELDS

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \mathrm{tr}\left(D_\mu\phi D^\mu\phi\right) - \frac{\lambda}{4}\left(v^2 - |\phi|^2\right)^2$$

$|\phi|^2 = {\phi^1}^2 + {\phi^2}^2 + {\phi^3}^2 = 2\,\mathrm{tr}\,\phi^2$





VACUUM MANIFOLD IS S^2 WITH RADIUS v

$L = T - V$

$$T = \int d^3x \left(\mathrm{tr}(\vec{E}\cdot\vec{E}) + \mathrm{tr}(D_0\phi D_0\phi)\right)$$

$$V = \int d^3x \left(\frac{1}{2}\mathrm{tr}(F_{ij}F_{ij}) + \mathrm{tr}(D_i\phi D_i\phi) + \frac{\lambda}{4}\left(v^2 - |\phi|^2\right)^2\right)$$

$B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$

$\phi = v t^3 \quad A_i = 0$ VACUUM $\quad SU(2) \longrightarrow U(1)$

$e^{i\alpha t^3}$

IN GENERAL THE GENERATOR $\alpha \vec{\phi}$ IS UNBROKEN (COMMUTES WITH $\vec{\phi}$)

PERTURBATIVE SPECTRUM:

$$\begin{cases} m_\gamma = 0 & U(1) \text{ PHOTON MASSLESS} \\ m_{W^\pm} = g v \\ m_{Higgs} = \sqrt{2\lambda}\, v & \text{HIGGS FIELD IS MASSLESS FOR } \lambda = 0 \text{ (NO POTENTIAL)} \end{cases}$$

REMEMBER THAT POTENTIAL WAS NOT NECESSARY TO HAVE STABILITY (DERRICK'S THEOREM)

$\lambda = 0$ CORRESPOND TO BPS CASE (THERE IS BOGOMOLN'Y DECOMPOSITION).

FINITE ENERGY CONDITION

$r \to \infty \qquad \phi : S^2 \longrightarrow S^2$

$r \to \infty$ SPHERE

VACUUM MANIFOLD

$n \in \pi_2 \left(S^2 \right) = \mathbb{Z}$ HOMOTOPY CLASSES

$\pi_2 \left({SU(2)}/{U(1)} \right) = \pi_1 \left(U(1) \right)$

E-L EQUATIONS:

$$\begin{cases} D_\mu D^\mu \phi = \lambda\left(v^2 - |\phi|^2\right)\phi \\ D_\mu F^{\mu\nu} = -ig\left[D^\nu \phi, \phi\right] \end{cases}$$

FOR STATIC SOLUTIONS

$$\begin{cases} D_i D_i \phi = -\lambda\left(v^2 - |\phi|^2\right)\phi \\ D_i F_{ij} = -g\left[D_j \phi, \phi\right] \end{cases}$$

NOW WE NEED AN ANSATZ. FOR THE 1-MONOPOLE SOLUTION WE CAN CHOOSE THE SO CALLED "HEDGEHOG" ANSATZ:

$$\begin{cases} \phi = v\,h(r)\,\dfrac{x^a}{r}\,t^a \\ A_i = \dfrac{1}{g}\left(1 - K(r)\right)\epsilon_{ija}\,\dfrac{x^j}{r^2}\,t^a \end{cases}$$

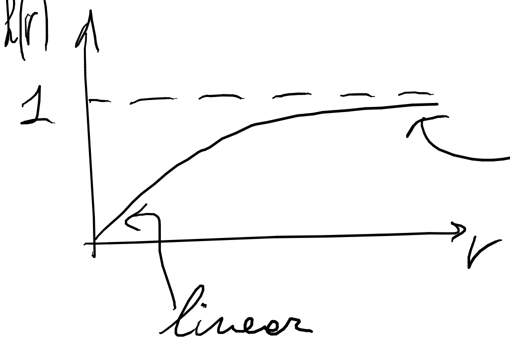
NOTE THAT WE "LOCKED" TOGETHER THE SPACE INDEX AND THE GAUGE INDEX.

$$\left(SU(2)_{GAUGE} \times SU(2)_{ROTATION} \xrightarrow{} SO(2)_{DIAGONAL}\right)$$

MONOPOLE SOLUTION.

AT INFINITY $\phi \to h(\infty)\,\frac{x^a t^a}{r}$

BY CHOOSING $h(\infty) = r$ WE HAVE EXACTLY THE [1] OF $\pi_2(S^2) = \mathbb{Z}$.



$\frac{e^{-m_\phi r}}{r}$

$h(0) = 0$

$h(\infty) = 1$

BOUNDARY CONDITIONS

$K(\infty) = 0 \qquad K(0) = 1)$

$$A_i \to \frac{1}{g}\epsilon_{ija}\frac{x^j}{r^2}t^a \qquad \phi \to \frac{x^a t^a}{r}$$

THIS IS REQUIRED FOR FINITE ENERGY. LET US CHECK:

$$D_i\phi = \partial_i\phi - ig[A_i,\phi]$$

$$= t^a\,\frac{\delta_{ia} - \hat{x}_i\hat{x}_a}{r} - i\frac{1}{r}\epsilon_{ija}\hat{x}^j\hat{x}^b[t^a,t^b]$$

$$= \frac{t^a}{r}(\delta_{ia} - \hat{x}_i\hat{x}_a) + \frac{1}{r}\underbrace{\epsilon_{ija}\epsilon_{abc}}_{\delta_{ib}\delta_{jc} - \delta_{ic}\delta_{jb}}\hat{x}^j\hat{x}^b t^c$$

$$= \frac{t^c}{r}(\delta_{ic} - \hat{x}_i\hat{x}_c) + \frac{1}{r}t^c(\hat{x}_i\hat{x}_c - \delta_{ic})$$

$$= 0\,!$$

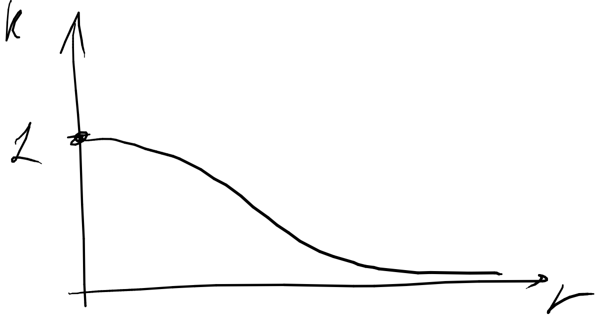
$$\partial_i x^a = \delta_i^a$$

$$\partial_i r = \hat{x}_i$$

$$\partial_i\hat{x}_j = \partial_i\frac{x_j}{r} = \frac{\delta_{ij}}{r} - \frac{x_j\hat{x}_i}{r^2} = \frac{\delta_{ij} - \hat{x}_j\hat{x}_i}{r}$$

$r \to 0 \quad \phi \propto x^a t^a + \mathcal{O}(x^2)$

$A \sim \mathcal{O}(1) \quad$ IF $k(0) = 1$



Monopole Magnetic Field

THE ASYMPTOTIC GAUGE FIELD $A_i \to \frac{1}{g} \epsilon_{ija} \frac{x_j t^a}{r^2}$

CONTAINS A PURE GAUGE PART THAT IS NEEDED TO MAKE $D_i \phi$ VANISH, BUT ALSO AN EXTRA TERM THAT IS NOT PURE GAUGE.

$$\partial_i \frac{x^j}{r^2} = \frac{\delta_{ij} - 2\hat{x}_i \hat{x}_j}{r^2}$$

$$F_{ij} = \partial_i A_j - \partial_j A_i - ig[A_i, A_j]$$

$$= \frac{1}{g}\left(\partial_i \epsilon_{jst} \frac{\hat{x}^s}{r} t^t - \partial_j \epsilon_{ist} \frac{\hat{x}^s}{r} t^t\right) - \frac{i}{g} \epsilon_{ist} \epsilon_{jpq} \frac{\hat{x}^s}{r} \frac{\hat{x}^p}{r} [t^t, t^q]$$

$$= \frac{t^t}{g}\left(\epsilon_{jst} \frac{\delta_{is} - 2\hat{x}_i \hat{x}_s}{r^2} - \epsilon_{ist} \frac{\delta_{js} - 2\hat{x}_j \hat{x}_s}{r^2}\right) \underbrace{- \frac{1}{g} \epsilon_{ist} \epsilon_{jpq} \epsilon_{tqr} t^r \frac{\hat{x}^s}{r} \frac{\hat{x}^p}{r}}_{-\frac{1}{g} \epsilon_{jpi} t^s \frac{\hat{x}^s \hat{x}^p}{r^2}}$$

THE MAGNETIC FIELD OF THE UNBROKEN $U(1)$ IS:

IF $\phi = v t^3$ $\quad f_{ij} = F^3_{ij} = 2 \frac{tr(F_{ij}\phi)}{v}$

$v = \sqrt{2\, tr\, \phi^2}$

SO AN INVARIANT DEFINITION OF THE ABELIA MAGNETIC FIELD IS

$$f_{ij} = \sqrt{2} \frac{tr(F_{ij}\phi)}{\sqrt{tr\phi^2}}$$

WE CAN COMPUTE IT FOR THE PREVIOUSLY OBTAINED F_{ij}

$$f_{ij} = 2\, tr\left(F_{ij}\, \hat{x}^a t^a\right)$$

LA PARTE CHE PROVIENE DA $\partial_i A_j - \partial_j A_i$ NON DA NESSUN CONTRIBUTO

$$f_{ij} = 2 \hat{x}^a \frac{\delta^{as}}{2} (-) \epsilon_{jpi} \frac{\hat{x}_s \hat{x}_p}{g r^2}$$

$$= -\frac{1}{g} \epsilon_{ijp} \frac{\hat{x}_p}{r^2}$$

$b_i = \frac{1}{2} \epsilon_{ijk} f_{jk} = -\frac{\hat{x}^i}{g r^2}$ $\quad$ MAGNETIC FLUX $= -\frac{4\pi}{g}$

(DIRAC QUANTIZATION WITH FUNDAMENTAL CHARGE $\frac{g}{2}$)

BPS MONOPOLE

(BOGOMOL'NY-PRASAD-SOMMERFELD)

$\lambda = 0$ IS THE CASE WHEN BOGOMOL'NY TRICK CAN BE USED.

NOTE THAT WE ARE SENDING $\lambda \to 0$ WHILE KEEPING $|\phi| = v$ FIXED. (THIS IS POSSIBLE IN $d=3$)

$$E = \int d^3x \left(\operatorname{tr}(\vec{B}\cdot\vec{B}) + \operatorname{tr}(\vec{D}\phi\cdot\vec{D}\phi) \right)$$

STATIC ENERGY
$V(\phi) = 0$

$$= \int d^3x \left(\operatorname{tr}(B_i + D_i\phi)^2 - 2\operatorname{tr}(B_i D_i \phi) \right)$$

$$\operatorname{tr}(B_i D_i \phi) = \partial_i \operatorname{tr}(B_i \phi) \qquad \text{IDENTITY}$$

$$\operatorname{tr}(\partial_i B\,\phi + B_i\,\partial\phi)$$

$$= \operatorname{tr}\Big(\underbrace{D_i B\,\phi}_{0} + ig\underbrace{[A_i, B_i]\phi + B_i D_i\phi + igB_i[A_i,\phi]}_{0} \Big)$$

0 BIANCHI IDENTITY
$D_i B_i = 0$

$$= \operatorname{tr}(B_i D_i \phi)$$

SO

$$E = \int d^3x \left(B_i + D_i\phi\right)^2 - \underbrace{\int d^3x \; 2\,\partial_i \operatorname{tr}(B\phi)}_{\text{TOTAL DERIVATIVE}}$$

$\Rightarrow$ MAGNETIC FLUX

$$\left.\begin{aligned} \operatorname{tr}(F_{ij}\phi) &= -\frac{v}{2g}\,\epsilon_{ijk}\,\frac{\hat{x}_k}{r^2} \\ \operatorname{tr}(B_i\phi) &= -\frac{v}{2g}\,\frac{\hat{x}_i}{r^2} \\ E &= \int d^3x \left(B_i + D_i\phi\right)^2 + \frac{4\pi v}{g} \end{aligned}\right\} \text{FOR 1 MONOPOLE}$$

BOGOMOLNY EQUATIONS:

$$B_i \pm D_i\phi = 0$$

BOGOMOLNY BOUND

$$E \geq \frac{4\pi v}{g}|n|$$

$\pi_2(S^2)$ TOPOLOGICAL SECTOR.

$$\boxed{M_{mon} = \frac{4\pi}{e^2} M_W}$$

UNLIKE THE SINE-GORDON CASE, THERE IS A MODULI SPACE OF SOLUTIONS

$$\dim_{\mathbb{R}} \mathcal{M}_n = 4n$$

4 FOR EVERY MONOPOLE CONSTITUENT

LET'S GO BACK TO THE HEDGEOG ANSATZ

$$\begin{cases} \phi = v\, h(r)\, \dfrac{x^a}{r}\, t^a \\ A_i = \dfrac{1}{g}\left(1-k(r)\right) \epsilon_{ija} \dfrac{x^j t^a}{r^2} \end{cases}$$

THE BPS EQUATION IS

$$B_i + D_i \phi = 0$$

$$D_i\phi = \partial_i \phi - i g\, [A_i, \phi] =$$

$$= v\, t^a \partial_i \left(h\, \hat{x}^a\right) - i v h \frac{x^b}{r}(1-k)\epsilon_{ija}\frac{x^j}{r^2}\,[t^a, t^b]$$

$$= v\left(t^a\left(h' \hat{x}^i \hat{x}^a + h\,\frac{\delta_{ia} - \hat{x}_i \hat{x}_a}{r}\right) + h(1-k)\frac{x^b}{r}\underbrace{\epsilon_{ija}\frac{x^j}{r^2}\epsilon^{abc}}_{\delta_{ib}\delta_{jc} - \delta_{ic}\delta_{jb}} t^c\right)$$

$$= v\left(t^a\left(h' \hat{x}^i \hat{x}^a + h\,\frac{\delta_{ia} - \hat{x}_i \hat{x}_a}{r}\right) + h(1-k)\frac{1}{r}\left(\hat{x}^i \hat{x}^a t^a - t^i\right)\right)$$

$$= v\left\{ t^i\left(\frac{h}{r} - \frac{h(1-k)}{r}\right) + \hat{x}^i \hat{x}^a t^a\left(h' - \frac{h}{r} + h(1-k)\frac{1}{r}\right)\right\}$$

$$D_i\phi = r\left\{ t^i \frac{hk}{r} + \hat{x}^i \hat{x}^a t^a \left(h' - \frac{hk}{r}\right)\right\}$$

$$B_i = \frac{1}{2} \epsilon_{ijk} F_{jk} = \frac{1}{2} \epsilon_{ijk}\left(\partial_j A_k - \partial_k A_j - ig\left[A_j, A_k\right]\right) =$$

$$= \epsilon_{ijk}\left(\partial_j A_k - \frac{i}{2} g \left[A_j, A_k\right]\right) =$$

$$= \epsilon_{ijk}\frac{1}{g}\left(\partial_j\left((1-k)\,\epsilon_{ksq}\frac{\hat{x}^s t^q}{r}\right) - \frac{i}{2}(1-k)^2 \epsilon_{jsq}\epsilon_{kwz}\frac{\hat{x}^s}{r}\frac{\hat{x}^w}{r}\left[t^q, t^z\right]\right)$$

$$= \epsilon_{ijk}\frac{1}{g}\left(\epsilon_{ksq}\, t^q\, \partial_j\left((1-k)\frac{\hat{x}^s}{r}\right) + \frac{1}{2}(1-k)^2\epsilon_{jsq}\epsilon_{kwz}\frac{\hat{x}^s \hat{x}^w}{r^2}\epsilon_{qzu}\, t^u\right)$$

$$= \frac{1}{g}\Bigg(\left(\delta_{is}\delta_{jq} - \delta_{iq}\delta_{js}\right) t^q \left(-k' \frac{\hat{x}^j \hat{x}^s}{r} + \frac{(1-k)}{r^2}\left(\delta_{js} - \hat{x}_j \hat{x}_s\right) + (1-k)\hat{x}^s\left(-\frac{1}{r^2}\right)\hat{x}^j\right)$$

$$+ \frac{1}{2}(1-k)^2\left(\delta_{ks}\delta_{iq} - \delta_{kq}\delta_{is}\right)\left(\delta_{ku}\delta_{wq} - \delta_{kq}\delta_{wu}\right) \cdot \frac{\hat{x}^s \hat{x}^w}{r^2}\, t^u\Bigg)$$

$$\left(\delta_{su}\delta_{iw} - \delta_{is}\delta_{wu} - \delta_{uw}\delta_{is} + 3\,\delta_{uw}\delta_{is}\right)$$

$$= \frac{1}{g}\Big\{ t^i \left(\frac{k'}{r} - \frac{(1-k)}{r^2} \right) + t^j \left(-\frac{k'}{r} \hat{x}^i \hat{x}^j + \frac{(1-k)}{r^2} \left(\delta_{ji} - 2 \hat{x}_i \hat{x}_j \right) \right)$$

$$+ \frac{1}{2} (1-k)^2 \left(\delta_{su} \delta_{iw} + \delta_{uw} \delta_{is} \right) \frac{\hat{x}^s \hat{x}^w}{r^2} t^u \Big\}$$

$$= \frac{1}{g}\Big\{ t^i \frac{k'}{r} + t^j \hat{x}^i \hat{x}^j \left(-\frac{k'}{r} - 2 \frac{(1-k)}{r^2} \right)$$

$$+ \frac{1}{2} (1-k)^2 \cdot \frac{2}{r^2} t^j \hat{x}^i \hat{x}^j \Big\}$$

$$= \frac{1}{g}\Big\{ t^i \frac{k'}{r} + t^j \hat{x}^j \hat{x}^i \left(-\frac{k'}{r} - \frac{1-k^2}{r^2} \right) \Big\}$$

FINALLY $\partial_i \phi = -B_i$:

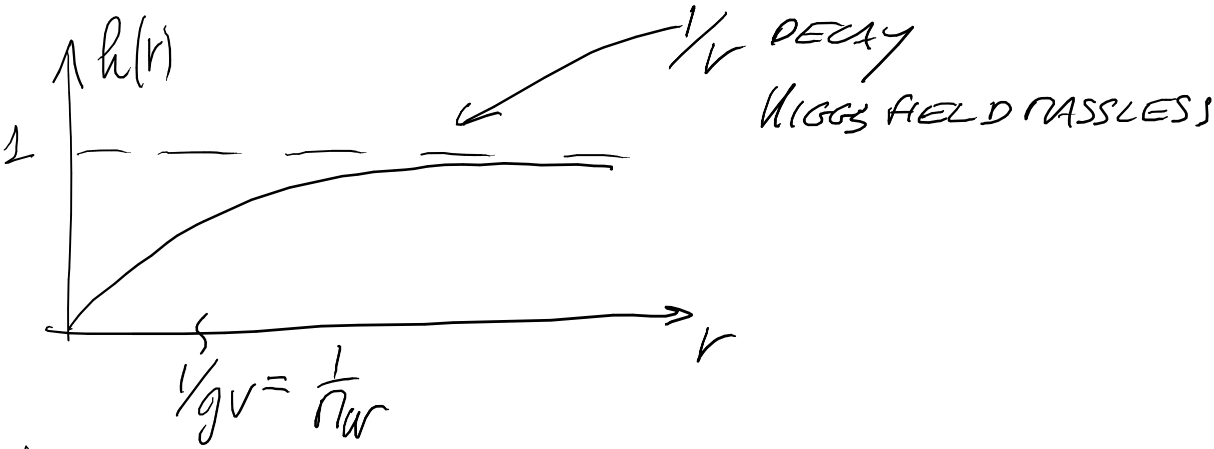
$$\begin{cases} r \dfrac{hk}{r} = -\dfrac{1}{g} \dfrac{k'}{r} \\ r \left(h' - \dfrac{hk}{r} \right) = +\dfrac{1}{g} \left(\dfrac{k'}{r} + \dfrac{1-k^2}{r^2} \right) \end{cases}$$

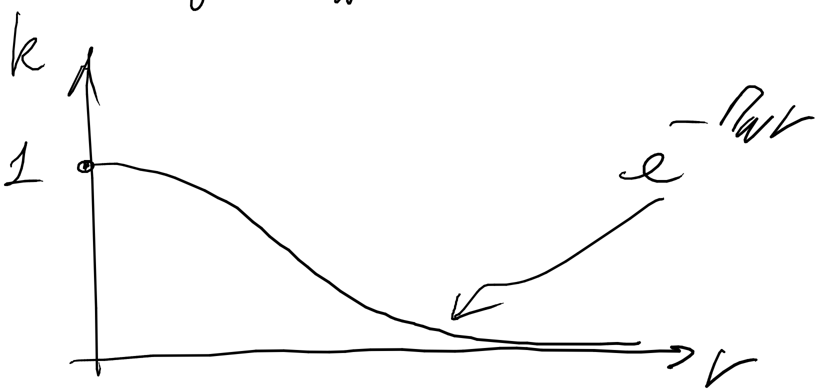
WE CAN RESCALE THE LENGTH $r \longrightarrow g v r$

$$\begin{cases} hk + k' = 0 \\ r^2 h' - 1 + k^2 = 0 \end{cases}$$

$$\begin{cases} h(r) = \coth(gvr) - \dfrac{1}{gvr} \\ k(r) = \dfrac{gvr}{\sinh(gvr)} \end{cases}$$

SOLUTION





BPS MONOPOLE EQUATIONS, UNLIKE BPS VORTICES, ARE INTEGRABLE.